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Generalized Clausius–Mossotti Formula for Random Composite with Circular Fibers

L. Berlyand[1] **and V. Mityushev**[2]

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An important area of materials science is the study of effective dielectric, thermal and electrical properties of two phase composite materials with very different properties of the constituents. The case of small concentration is well studied and analytical formulas such as Clausius–Mossotti (Maxwell–Garnett) are successfully used by physicists and engineers. We investigate analytically the case of an arbitrary number of unidirectional circular fibers in the periodicity cell when the concentration of the fibers is not small, i.e., we account for interactions of all orders (pair, triplet, etc.). We next consider transversely-random unidirectional composite of the parallel fibers and obtain a closed form representation for the effective conductivity (as a power series in the concentration v). We express the coefficients in this expansion in terms of integrals of the elliptic Eisenstein functions. These integrals are evaluated and the explicit dependence of the parameter d, which characterizes random position of the fibers centers, is obtained. Thus we have extended the Clausius–Mossotti formula for the non dilute mixtures by adding the higher order terms in concentration and qualitatively evaluated the effect of randomness in the fibers locations. In particular, we have proven that the periodic array provides extremum for the effective conductivity in our class of random arrays ("shaking" geometries). Our approach is based on complex analysis techniques and functional equations, which are solved by the successive approximations method.

KEY WORDS: Effective conductivity; homogenization; random composite material; functional equation.

[1] Department of Mathematics, Penn State University, University Park, Pennsylviana 16802 and Department of Mathematics and Computer Science, University of Akron, Akron, Ohio 44325-4002; e-mail: berlyand@math.psu.edu

[2] Department of Mathematics, Pedagogical University, Ul. Arciszewskiego 22 B, Slupsk 76-200, Poland; e-mail: mityu@wsp.slupsk.pl

1. INTRODUCTION

One of the most important problem of materials science is analysis of the various fields and prediction of the effective properties of composite materials. Such problems can be studied by modeling of the processes by Laplace's equations with appropriate interface conditions. The solution of this boundary value problem determines the physical field (e.g., electric potential) in the composite and the effective conductivity tensor Λ_e, which describes overall transport properties of a composite. In this paper we consider composites which consist of a uniform background (matrix or host) reinforced by unidirectional cylindrical fibers. This mathematical model describes heat conduction, electric and dielectric transport properties, permeability of a liquid flow and anti-plane elastic problems. In order to obtain a formula for Λ_e it is sufficient to solve a boundary value problem in the unit periodicity cell.[2, 9] Grigolyuk and Filshtinskij[12] used the method of integral equations to solve two-dimensional cell problems numerically. Main focus of this work was directed towards the elastic problems. Another constructive approach is based on the reduction to an infinite systems of linear algebraic equations, which can be solved numerically by the truncation procedure. Berdichevskij[4] used series to obtain approximations for Λ_e in the case of one inclusion in a three-dimensional periodic cell. In this work a three-dimensional analog of the elliptic Weierstrass' functions had been developed. The application of Rayleigh's method[17] reduces the two-dimensional conductivity problem to an infinite system of linear algebraic equations. The latter can be truncated and solved numerically. McPhedran *et al.* in ref. 19 have extended Rayleigh's techniques (in particular, derived an expression for the multipole coefficients) and obtained approximate formulas for the effective conductivity of the square and hexagonal arrays of cylinders. Similar method was applied by Sangani and Yao.[28] Using special derivative operators they reduced the problem to an infinite system of linear algebraic equations. Analytic properties of Λ_e as a function of the contrast parameter have been exploited by Bergman and Dunn[5] and Milton[20] to construct bounds for Λ_e. These bounds provide an efficient tool for the numerical evaluation of Λ_e. Cheng and Greengard[7] studied problems for composite materials with a finite (large) number of inclusions in the plane. The method of integral equations and the method of images have been used to reduce the problem to an infinite system of linear algebraic equations. A number of interesting numerical results have been presented in this work.

A simple formula on Λ_e is given by the famous Clausius–Mossotti approximation (for history and applications see refs. 16 and 20). In two dimensions under the assumption that the composite is macroscopically

isotropic, that is $\Lambda_e = \begin{pmatrix} \lambda_e & 0 \\ 0 & \lambda_e \end{pmatrix}$, this formula can be written as follows:

$$\lambda_e = \frac{1+\rho v}{1-\rho v} + O(v^2) \tag{1.1}$$

Here $\rho = (\lambda_1 - 1)/(\lambda_1 + 1)$, the conductivity of the matrix is equal to unity ($\lambda = 1$), the conductivity of the fibers is λ_1 and the concentration v tends to zero. Formula (1.1) provides a good approximations for λ_e in the case of circular unidirectional fibers for small v (the dilute limit). Many authors (see above citations) improved the formula (1.1) by considering special geometrical structures such as the square and the hexagonal arrays. As it was noted above they reduced the problem to the integral equations or to the infinite systems of linear algebraic equations, and then used numerics. A method of functional equations has been developed in refs. 23–26 for the planar interface conditions (conjugation conditions). At first the functional equations appeared in Golusin's paper.[11] In particular, the method of functional equations has been applied to improve (1.1) by evaluation of the next terms in the expansion in ρ. For the sake of completeness this result is presented in Theorems 3.2 and 3.3 of the present paper. One can see that these terms contain geometrical parameters such as positions of the cylinders and their radii. For instance, the following formula for the effective conductivity tensor $\Lambda_e = \begin{pmatrix} \lambda_e^x & \lambda_e^{xy} \\ \lambda_e^{xy} & \lambda_e^y \end{pmatrix}$ follows from Theorem 3.3

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho \sum_{m=1}^{N} v_m + \frac{\rho^2}{\pi} \sum_{k \neq m} v_m v_k E_2(a_k - a_m) + O(\rho^3) \tag{1.2}$$

as $\rho \to 0$, where N is the number of inclusion in the unit cell, a_m are the centers of inclusions (complex numbers), $v_m = \pi r_m^2$, r_m are the radii of inclusions, E_2 is the Eisenstein function (see Appendix C). Equation (1.2) allows us to calculate the components λ_e^x and λ_e^{xy} of the tensor Λ_e, since (1.2) is a complex equality and is equivalent to two real ones. The component λ_e^y is calculated by similar way.

In this paper we solve functional equations using a power series in concentration v. Hence, we improve (1.1) by explicit evaluation of the higher order terms *in the expansion in v for arbitrary fixed* ρ (unlike the previous work,[23] where the expansion in ρ have been considered). In Theorem 3.4 we derive a simple algorithm based on successive approximations to evaluate the flux in the composite material. Using these formulas for the local fields we obtain formula (4.9) for the effective conductivity. Higher-order terms explicitly computed through the geometrical parameters: centers and radii of the cylinders (Section 4, (4.14)–(4.18)). The series

in ρ is rather convenient to obtain convergence results for the method of functional equations. The series in v (e.g., (4.10)) is convenient to obtain higher order constructive formulas for the effective conductivity, since such a series explicitly contains elliptic functions in corrections of all orders (also compare the expansions (3.10)–(3.11) and (3.8); in the latter the higher order terms in the expansion in ρ have been expressed through the double operator's series). The Eisenstein functions are well studied, high precision fast numerical algorithms and analytical formulas based on θ-functions[13, 31] are readily available for computing the elliptic functions. To summarize, we remark that the series in v corresponds to the physical nature of the problem, in particular, it can be used for the high contrast case (compare with (1.2), where ρ tends to zero), when $|\rho|$ is closed to 1 and the expansion in v provides simple approximate formulas for the higher order terms in the expansion of λ_e. Note that the effective conductivity λ_e is an analytic function in v in the 2D case. The latter is not so in the 3D case (see, for example, refs. 18 and 4).

We next consider a random distribution of the fibers. To simplify presentation we consider the unit cell with identical inclusions whose centers a_1, a_2, a_3, a_4 are random variables. Each center is uniformly distributed in a disk of the radius d. Centers of these disks form a square array on the plane (shaded disks, see Fig. 1) whereas the disks which correspond to the fiber's cross-sections do not form the periodic array. Hence, we

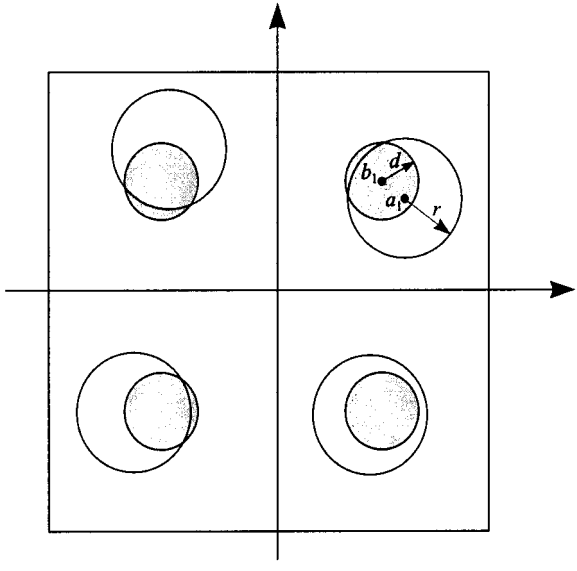

Fig. 1. "Shaking" geometry.

investigate a random "shaking" of the fibers about the periodic square array (d does not have to be small and therefore our results are not perturbative in d). We represent the expected value $\langle \lambda_e \rangle$ of the random value λ_e in the form $\langle \lambda_e \rangle = \lambda_e^{(0)} + \langle \tilde{\lambda}_e \rangle$ where $\lambda_e^{(0)}$ corresponds to the deterministic structure, when $d=0$ and the centers are located at the sites of the regular square array; the term $\langle \tilde{\lambda}_e \rangle$ appears due to the randomness. The parameter d has been chosen so that the fibers cannot touch. However, our method also works when cylinders touch each other but do not form a spanning cluster of touching cylinders in the periodic cell, i.e., no infinite chains of touching disks in the plane are allowed (see also Remark 3.1 from Section 3). In other words we do not consider the percolation phenomenon. The quantity $\langle \tilde{\lambda}_e \rangle$ is a correction due to the random shaking. We represent $\langle \tilde{\lambda}_e \rangle$ as a power series in v and evaluate coefficients in this series in terms of the integrals of the elliptic Eisenstein functions (see Appendix C and D). We show that these integrals are analytic in d (see formulas (5.4) and (6.20)). Thus we have extended the Clausius–Mossotti formula (1.1) for a non dilute case by adding the higher order terms in concentration v and qualitatively evaluated the effect of randomness by evaluating the dependence on the "shaking" parameter d (see (5.1), (5.4)). Moreover, we have proved that the correction $\langle \tilde{\lambda}_e \rangle$ is positive, if $\lambda_1 > \lambda = 1$, i.e., $\rho > 0$. This leads us to an important qualitative conclusion: any random shaking described above (see Fig. 1) of the periodic square array (see Fig. 2) results in the increase of the effective conductivity. In

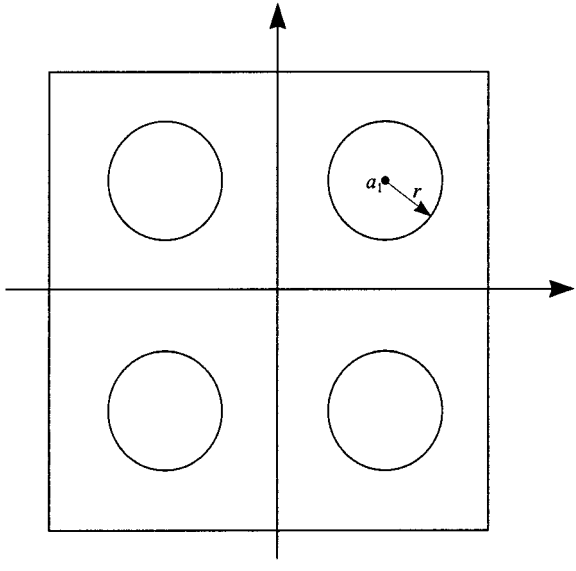

Fig. 2. The periodic array.

other words the periodic array provides minimum of the effective conductivity in our class of arrays (the "shaking" geometries). We mention here a heuristic argument which suggests that the shaking geometries provide a reasonable approximation for random mixtures at high concentrations. Indeed there is not much room for the inclusions to move around, when their density is relatively high. Therefore the inclusions could naturally form some kind of a shaking pattern, whereas at small concentrations the random patterns could be very different from the shaking geometries.

We also remark that many authors studied various aspects of homogenization for random composite materials (see, for example, refs. 14, 27, 29, 30, and 6). Our focus is to look for special cases where explicit analytical formulas for the effective conductivity can be obtained. We also note that a generic approach based on the *n-point* probability functions and variational bounds have been developed and used by several authors (see refs. 30, 21, 3, 20 and references therein).

Finally we remark that in the paper[15] the extremal property of the periodic array has been established for the case of the dilute limit (small concentration of the inclusions) in 3D. This was done under an assumption that the centers of the square array of small balls can be randomly perturbed (a small perturbation). Of course, the case of small concentrations is quite different from our consideration since we allow arbitrary high concentrations up to the touching (touching excluded) and therefore we account for interactions of all orders (unlike the dilute limit case when the interactions are negligible). Moreover our "shaking" parameter d, which characterizes random perturbation of the centers of disks is not small, that is we allow all perturbations up to the touching (touching excluded). In this regard the small perturbation result of ref. 15 can be viewed as a liner part (approximation or expansion) in d of our formula for the effective conductivity. In other words the result of ref. 15 has a local nature (small concentration, and small random perturbation about the periodic array). Our result shows that the conclusion of the Kozlov's theorem[15] also holds for the global case (non small concentration and non small d).

2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

We consider a lattice $\mathcal{Q}$ which is defined by the two fundamental translation vectors 1 and i ($i^2=-1$) in the complex plane $\mathbf{C}\cong\mathbf{R}^2$ of the complex variable $z=x+iy$. The zeroth cell $Q_{(0,0)}$ (the basis of $\mathcal{Q}$) is the square $\{z=t_1+it_2\in\mathbf{C}: -1/2<t_p<1/2,\ p=1,2\}$. Let $\mathcal{E}:=\bigcup_{m_1,m_2}\{m_1+im_2\}$ is the set of the lattice points, where m_1 and m_2 belong to the set of integer numbers $\mathbf{Z}$. The lattice $\mathcal{Q}$ consists of the cells $Q_{m_1,m_2}=Q_{(0,0)}+m_1+im_2:=\{z\in\mathbf{C}: z-m_1-im_2\in Q_{(0,0)}\}$. Let us consider mutually disjoint equal disks

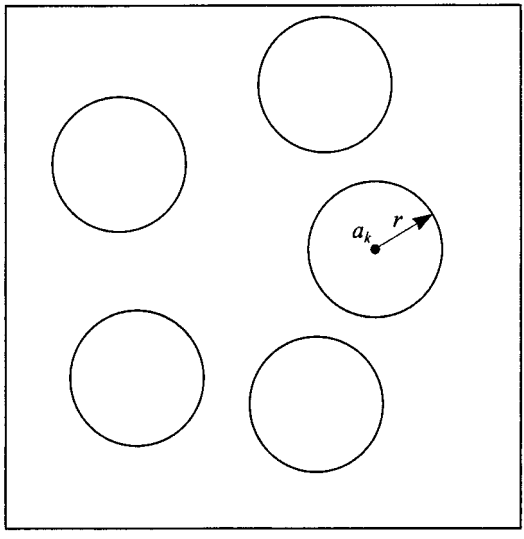

Fig. 3. The periodicity cell with inclusions.

(inclusions) $D_k := \{z \in \mathbf{C} : |z - a_k| < r\}$ $(k = 1, 2, ..., N)$ in the zeroth cell $Q_{(0,0)}$ (see Fig. 3).

Let $T_k := \{t \in \mathbf{C} : |t - a_k| = r\}$ be the boundary of D_k, and $\mathscr{D}_0 := Q_{(0,0)} \setminus (\bigcup_{k=1}^{N} D_k \cup T_k)$ be the connected phase (the matrix or host) in the zeroth cell $Q_{(0,0)}$. Hereafter we use the letters z and t for a variable inside the domain, and on the boundary of the domain respectively. We study the effective conductivity of the doubly periodic composite material, when the domains $\mathscr{D}_0$ and D_k in the cell $Q_{(0,0)}$ are occupied by materials of conductivities $\lambda = 1$ and $\lambda_1 > 0$, respectively (Fig. 3), i.e., the unbounded domain $\mathscr{D}_{\text{per}} := \bigcup_{m_1, m_2} (\mathscr{D}_0 \cup \partial Q_{(0,0)} + m_1 + im_2)$ and the disks $D_k + m_1 + im_2$ are occupied by two different materials and generate a composite material with a doubly periodic planar structure. The potential $u(z)$ satisfies the Laplace equation

$$\Delta u = 0 \qquad \text{in} \quad \bigcup_{k=1}^{N} (D_k + m_1 + im_2) \cup (\mathscr{D}_0 + m_1 + im_2) \tag{2.1}$$

with the conjugation conditions:

$$u^+ = u^-, \qquad \frac{\partial u^+}{\partial n} = \lambda_1 \frac{\partial u^-}{\partial n} \qquad \text{on } T_k, \quad k = 1, 2, ..., N \tag{2.2}$$

where $\partial/\partial n$ is the outward normal derivative and

$$u^+(t) := \lim_{z \to t,\, z \in \mathscr{D}_0} u(z) \qquad u^-(t) := \lim_{z \to t,\, z \in D_k} u(z)$$

$t \in T_k$, $k = 1, 2, \ldots, N$. We also impose on $u(z)$ the quasi periodicity conditions, i.e., $u(z)$ has constant jumps along the unit cell. We consider the external field applied in the x-direction given by the following conditions:

$$u(z+1) = u(z) + 1, \qquad u(z+i) = u(z) \tag{2.3}$$

We introduce the complex potentials $\varphi(z)$ and $\varphi_k(z)$ analytic in $\mathscr{D}_0$ and D_k, respectively, and continuously differentiable in the closures of $\mathscr{D}_0$ and D_k. The harmonic and analytic functions are related by the equalities

$$u(z) = \begin{cases} \Re(\varphi(z) + z), & z \in \mathscr{D}_0 \\ \dfrac{2}{1+\lambda_1}\, \Re\varphi_k(z), & z \in D_k, \quad k = 1, 2, \ldots, N \end{cases} \tag{2.4}$$

where $\Re$ stands for the real part. The normalization factor $2/(1+\lambda_1)$ is introduced, otherwise it would of appear in (2.6). Note that $\varphi(z)$ is single-valued (see Appendix B) in $\mathscr{D}_0$ and periodic in $\mathbf{C}$, namely (2.3) implies

$$\varphi(z+1) = \varphi(z) = \varphi(z+i) \tag{2.5}$$

Two real conditions (2.2) can be written in terms of the complex potentials

$$\varphi(t) = \varphi_k(t) - \rho\,\overline{\varphi_k(t)} - t, \quad |t - a_k| = r, \qquad k = 1, 2, \ldots, N \tag{2.6}$$

In Appendix B it is shown that (2.6) is equivalent to (2.2) up to an additive pure imaginary constant. In order to determine the current $\nabla u(x, y)$ we need only the derivatives

$$\begin{aligned} \psi(z) &:= \frac{\partial \varphi}{\partial z} = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}, && z \in \mathscr{D}_0 \\ \psi_k(z) &:= \frac{\partial \varphi_k}{\partial z} = \frac{\lambda_1 + 1}{2}\left(\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\right), && z \in D_k \end{aligned} \tag{2.7}$$

Differentiating (2.6) we arrive at the following problem

$$\psi(t) = \psi_k(t) + \rho\left(\frac{r}{t - a_k}\right)^2 \overline{\psi_k(t)} - 1, \qquad |t - a_k| = r, \quad k = 1, 2, \ldots, N \tag{2.8}$$

(see Appendix A for details).

3. METHOD OF FUNCTIONAL EQUATIONS

In this section we describe the method of functional equations (see refs. 11 and 26). We first provide a short overview of this method. In the above problem we have N contours T_k and N conjugation conditions on each contour, and we need to find $N+1$ functions $\psi, \psi_1,..., \psi_N$. Thus, roughly speaking, we need one more condition to close up the system. To get this condition we modify $\psi, \psi_1,..., \psi_N$ in their domains so that the obtained function $\Phi(z)$ (see (3.5)) is defined everywhere in the unit cell $Q_{(0,0)}$ (including the holes D_k) and its jump across each T_k is zero. The latter implies that $\Phi(z)$ is analytic inside the periodicity cell $Q_{(0,0)}$. Since it is double periodic we apply Liouville's theorem (for double periodic functions, see ref. 13) and conclude that $\Phi(z) = c$.

Let us consider the Banach space C_k of the functions continuous on T_k with the norm $\|\psi_k\| := \max_{T_k} |\psi_k(t)|$ $(k = 1, 2,..., N)$. Next we consider the closed subspaces $C_k^+ \subset C_k$ for which the functions ψ_k have analytic continuation into D_k. We also introduce the Banach space C^+ consisting of the functions $\Psi(t) := \psi_k(t) \in C_k^+$ for all $k = 1, 2,..., N$ with the norm $\|\Psi\| := \max_k \|\psi_k\|$ ($\Psi(t)$ is defined in all disks D_k). Let us introduce the Möbius transformations in $\bar{z}$

$$\alpha(\bar{z}) := \frac{r^2}{\overline{z - a_k - m_1 - im_2}} + a_k$$

for fixed $k = 1, 2,..., N$. If $m_1 = m_2 = 0$, then $\alpha(\bar{z})$ becomes the inversion with respect to the circle T_k. The right-hand side of the following relation

$$\overline{\psi_k(t)} = \overline{\psi_k\left(\frac{r^2}{\overline{t - a_k}} + a_k\right)}, \qquad t \in T_k$$

provides analytical continuation from $|t - a_k| = r$ into $|z - a_k| > r$. If $m_1 + im_2 \neq 0$, then $\alpha(\bar{z})$ consists of the inversion with respect to the circle $T_k + m_1 + im_2$ and translation by the vector $-(m_1 + im_2)$. One can see that $\alpha(\bar{z})$ with $m_1 + im_2 \neq 0$ transforms the closed disk $|z - a_k| \leqslant r$ into another closed disk which lies inside the open disk $|z - a_k| < r$. The latter geometrical property (contraction property) of $\alpha(\bar{z})$ allows us to assert that the operator

$$(W_{m_1, m_2 k}\psi_k)(z) = \left(\frac{r}{z - a_k - m_1 - im_2}\right)^2 \overline{\psi_k\left(\frac{r^2}{\overline{t - a_k - m_1 - im_2}} + a_k\right)} \tag{3.1}$$

is compact from C_k^+ into C_k^+ for $m_1 + im_2 \neq 0$ (there is no summation in the repeated index k). Indeed, applying the Cauchy integral formula[1] we have the following representation

$$\psi_k\left(\frac{r_2}{\overline{z-a_k-m_1-im_2}}+a_k\right)=\frac{1}{2\pi i}\int_{T_k}\frac{\psi_k(\tau)\,d\tau}{\tau-\dfrac{r^2}{\overline{z-a_k-m_1-im_2}}-a_k} \tag{3.2}$$

Thus (3.1) is a composition of a compact operator (3.2), the operator of the complex conjugation, and a bounded operator of the multiplication by $(r/(z-a_k-m_1-im_2))^2$. Therefore the operator (3.1) is bounded in $|z-a_k| \leqslant r$ $(m_1+im_2\neq 0)$. Compactness of (3.2) follows from the fact that its kernel is a continuous function in $(\tau, z)\in T_k\times D_k$. The latter follows from the contraction property. Analogously the operator W_{m_1,m_2k} for each m_1, m_2 and each $m=1, 2,..., N$; $m\neq k$ is compact and maps C_k^+ into C_m^+.

Remark 3.1. In the case of touching cylinders the operator (3.1) is not compact, blot it is bounded, if the touching disks do not generate an infinite chain in the plane. If the touching disks do not generate an infinite chain in the plane, then the convergence of the successive approximations still holds[26] and Theorems 3.3 and 3.4 hold.

We shall use W_{m_1,m_2k} to reduce the conjugation problem (2.8) to a system of functional equations. First we sum up $(W_{m_1,m_2k}\psi_k)(z)=(W_{(0,0)k}\psi_k)(z-m_1-im_2)$ over all translations m_1+im_2 to obtain a double periodic functions. To this end we expand $\psi_k(z)$ into the Taylor series

$$\psi_k(z)=\sum_{l=0}^{\infty}\psi_{lk}(z-a_k)^l$$

Next we use the following theorem.[23, 24]

Theorem 3.2. (i) The series $\sum_j (W_{jk}\psi_k)(z)$ $(j=(m_1,m_2), k$ fixed) converges absolutely and uniformly in the perforated cell $\mathscr{D}_0\cup\partial\mathscr{D}_0$. It defines a function which is analytic in $\mathscr{D}_0$, continuous in $\mathscr{D}_0\cup\partial\mathscr{D}_0$ and doubly periodic. This function can be written in the form

$$\sum_j (W_{jk}\psi_k)(z)=\sum_{l=0}^{\infty}\bar{\psi}_{lk}r^{2(l+1)}E_{l+2}(z-a_k),\qquad z\in\mathscr{D}_0 \tag{3.3}$$

where $E_l(z)$ is the Eisenstein function of order l (see Appendix C).

(ii) The series

$$\sum_j{}' (W_{jk}\psi_k)(z) := \sum_j (W_{jk}\psi_k)(z) - \left(\frac{r}{z-a_k}\right)^2 \overline{\psi_k\left(\frac{r^2}{\overline{z-a_k}} + a_k\right)}$$

defines a function analytic in the unit cell $Q_{(0,0)}$, continuous in $Q_{(0,0)} \cup \partial Q_{(0,0)}$. This function can be written in the form

$$\sum_j{}' (W_{jk}\psi_k)(z) = \sum_{l=0}^{\infty} \overline{\psi_{lk}}\, r^{2(l+1)} \sigma_{l+2}(z-a_k) \tag{3.4}$$

where $\sigma_l(z)$ is the modified Eisenstein function (see Appendix C).

(iii) The linear operator $\sum_j' W_{jk}\psi_k(z)$ is compact in C_k^+.

This theorem allows us to introduce an auxiliary function

$$\Phi(z) = \begin{cases} \psi_m(z) - \rho \sum_{k=1}^{N} \sum_{m_1, m_2}^{*} (W_{m_1, m_2 k}\psi_k)(z) - 1, & |z-a_m| \leqslant r_m, \\ \qquad m = 1, 2, \ldots, N & \\ \psi(z) - \rho \sum_{k=1}^{N} \sum_{m_1, m_2} (W_{m_1, m_2 k}\psi_k)(z), & z \in \mathscr{D}_0 \end{cases} \tag{3.5}$$

where $\sum_{k=1}^{N} \sum_{m_1, m_2}^{*} W_{m_1, m_2 k} := \sum_{k \neq m} \sum_{m_1, m_2} W_{m_1, m_2 k} + \sum_{m_1, m_2}' W_{m_1, m_2 m}$. It follows from Theorem 3.2 that $\Phi(z)$ is analytic in $\mathscr{D}_0$ and D_k ($k = 1, 2, \ldots, N$); it is also doubly periodic. Let us calculate the jump of $\Phi(z)$ across T_m

$$\Delta := \Phi^+(t) - \Phi^-(t) = \psi(t) - \rho \left(\frac{r}{t-a_m}\right)^2 \overline{\psi_m(t)} + 1, \qquad |t-a_m| = r$$

$$\Delta := \Phi^+(t) - \Phi^-(t) = \psi(t) - \rho \left(\frac{r}{t-a_m}\right)^2 \overline{\psi_m(t)} - \psi_m(t) + 1, \qquad |t-a_m| = r$$

Taking into account (2.8) we obtain $\Delta = 0$. Using the principle of analytic continuation and the generalized Liouville theorem for doubly periodic functions[13] we conclude that $\Phi(z) \equiv c$, where c is a constant. It will be shown in Section 4 that $c = 0$. Writing the relation $\Phi(z) = 0$ in $|z-a_m| \leqslant r$ we obtain the system of linear functional equations

$$\psi_m(z) = \rho \sum_{k=1}^{N} \sum_{m_1, m_2}^{*} (W_{m_1, m_2 k}\psi_k)(z) + 1, \qquad |z-a_m| \leqslant r_m, \quad m = 1, 2, \ldots, N \tag{3.6}$$

with respect to $\psi_m \in C_m^+$. The system (3.6) can be considered as an equation for the function $\Psi(z)$ in the space C^+

$$\Psi(z) = \rho \sum_{k=1}^{N} \sum_{m_1, m_2}^{*} W_{m_1, m_2 k} \Psi(z) + 1, \qquad z \in \bigcup_{m=1}^{N} (D_m \cup T_m) \tag{3.7}$$

where $\Psi(z) = \psi_m(z)$ in $|z - a_m| \leqslant r$ for each $m = 1, 2, ..., N$. We will use the following theorem:[23]

Theorem 3.3. Equation (3.7) has the unique solution in C^+. It can be found by the method of successive approximations, which gives the following series

$$\psi_m(z) = 1 + \rho \sum_{k_1=1}^{N} \sum_{j_1}^{*} W_{j_1 k_1} 1(z) + \rho^2 \sum_{k_1} \sum_{k_2} \sum_{j_1}^{*} \sum_{j_2}^{*} W_{j_1 k_1} W_{j_2 k_2} 1(z) + \cdots,$$
$$|z - a_m| \leqslant r_m, \qquad m = 1, 2, ..., N \tag{3.8}$$

The function $\psi(z)$ has the form

$$\psi(z) = \rho \sum_{k=1}^{N} \sum_{j} (W_{jk} \psi_k)(z), \qquad z \in \mathscr{D}_0 \cup \partial \mathscr{D}_0 \tag{3.9}$$

Convergence in C^+ means uniform convergence, which preserves analyticity in the limit. The operator $W_{m_1, m_2 k}$ depends analytically on r^2. Hence we can consider $\psi_m(z)$ determined by (3.8) as an analytic function in variable (ρ, r^2) in the bidisk $U \times V$, where U is the unit disk, V is the disk of the radius r_0^2. The critical radius r_0 is the minimum radius of the disks with the given set of the centers a_k $(k = 1, 2, ..., N)$ for which at least two disks are touching. This allows us to propose another method to solve (3.6) or (3.7). Namely we look for $\psi_m(z)$ in the form of series expansion in r^2:

$$\psi_m(z) = \psi_m^{(0)}(z) + r^2 \psi_m^{(1)}(z) + r^4 \psi_m^{(2)}(z) + \cdots \tag{3.10}$$

Using the representations (3.3) and (3.4) which is based on the Eisenstein functions and (3.10) we rewrite Eqs. (3.6) in the form

$$\sum_{s=0}^{\infty} r^{2s} \psi_m^{(s)}(z) = \rho \left[\sum_{k \neq m}^{N} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \overline{\psi_{lk}^{(s)}} r^{2(s+l+1)} E_{l+2}(z - a_k) \right.$$
$$\left. + \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \overline{\psi_{lm}^{(s)}} r^{2(s+l+1)} \sigma_{l+2}(z - a_m) \right] + 1,$$
$$|z - a_m| \leqslant r_m, \quad m = 1, 2, ..., N$$

where each term in (3.10) is expanded into the Taylor series

$$\psi_k^{(s)}(z) = \sum_{l=0}^{\infty} \psi_{lk}^{(s)} (z - a_k)^l$$

Collecting the coefficients of the like powers of r^2 we arrive at the following theorem:

Theorem 3.4. Let $u(z)$ be the solution of the cell problem (2.1)–(2.3) and let ∇u be the flux defined in (2.7):

$$\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \begin{cases} \psi(z) + 1, & z \in \mathscr{D}_0 \cup \partial \mathscr{D}_0 \\ \dfrac{2}{\lambda_1 + 1} \psi_k(z), & z \in D_k \cup T_k, \quad k = 1, 2, \ldots, N \end{cases}$$

Here $z = x + iy$, $\psi_k(z)$ and $\psi(z)$ are given by (3.10) and (3.9), respectively, where $\psi_m^{(p)}(z)$ are given by the following recurrence relations:

$$\begin{aligned}
\psi_m^{(0)}(z) &= 1 \\
\psi_m^{(1)}(z) &= \rho \left[\sum_{k \neq m}^{N} \overline{\psi_{0k}^{(0)}} E_2(z - a_k) + \overline{\psi_{0m}^{(0)}} \sigma_2(z - a_m) \right] \\
&= \rho \left[\sum_{k \neq m}^{N} E_2(z - a_k) + \sigma_2(z - a_m) \right] \\
\psi_m^{(2)}(z) &= \rho \left\{ \sum_{k \neq m}^{N} [\overline{\psi_{0k}^{(0)}} E_2(z - a_k) + \overline{\psi_{1k}^{(0)}} E_3(z - a_k) \right] \\
&\quad + \overline{\psi_{0m}^{(1)}} \sigma_2(z - a_m) + \overline{\psi_{1m}^{(1)}} \sigma_3(z - a_m) \Big\} \\
&\vdots \\
\psi_m^{(p+1)}(z) &= \rho \left\{ \sum_{k \neq m}^{N} \overline{\psi_{pk}^{(0)}} E_{p+2}(z - a_k) + \overline{\psi_{pm}^{(0)}} \sigma_{p+2}(z - a_m) \right. \\
&\quad + \sum_{k \neq m}^{N} \overline{\psi_{p-1,k}^{(1)}} E_{p+1}(z - a_k) + \overline{\psi_{p-1,m}^{(1)}} \sigma_{p+1}(z - a_m) + \cdots \\
&\quad \left. + \sum_{k \neq m}^{N} \overline{\psi_{0k}^{(p)}} E_2(z - a_k) + \overline{\psi_{0m}^{(p)}} \sigma_2(z - a_m) \right\}, \qquad p = 0, 1, 2, \ldots
\end{aligned} \tag{3.11}$$

Thus we have found a simple recursive algorithm which defines the flux ∇u (u is a solution of (2.1)–(2.3)) in terms of the Eisenstein functions, the contrast parameter ρ, and the radius r of the identical disks. We remark that in order to determine the effective conductivity it is sufficient to know the flux ∇u.

4. EFFECTIVE CONDUCTIVITY

Let us find the effective conductivity tensor

$$\Lambda_e = \begin{pmatrix} \lambda_e^x & \lambda_e^{xy} \\ \lambda_e^{xy} & \lambda_e^y \end{pmatrix} \tag{4.1}$$

of the composite material represented by the zero cell $Q_{(0,0)}$. We will compute λ_e^x and λ_e^{xy}, then λ_e^y can be calculated similarly to λ_e^x. The variational definition of Λ_e can be found in refs. 2 and 9. We will use an equivalent definition which can be obtained by a simple application of Green's formula

$$\lambda_e^x = \int_{\mathscr{D}_0} \frac{\partial u}{\partial x}\, dx\, dy + \lambda_1 \sum_{m=1}^{N} \int_{D_m} \frac{\partial u}{\partial x}\, dx\, dy \tag{4.2}$$

$$\lambda_e^{xy} = \int_{\mathscr{D}_0} \frac{\partial u}{\partial y}\, dx\, dy + \lambda_1 \sum_{m=1}^{N} \int_{D_m} \frac{\partial u}{\partial y}\, dx\, dy \tag{4.3}$$

where u is the solution of the problem (2.1)–(2.3). Using another Green's formula

$$\int_G \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\, dy = \int_{\partial G} f\, dx + g\, dy \tag{4.4}$$

we obtain

$$\int_{\mathscr{D}_0} \frac{\partial u}{\partial x}\, dx\, dy = \int_{\partial \mathscr{D}_0} u\, dy = \int_{\partial \mathscr{Q}_{(0,0)}} u\, dy - \sum_{m=1}^{N} \int_{\partial D_m} u\, dy \tag{4.5}$$

Here we assume that the curves $\partial \mathscr{Q}_{(0,0)}$ and ∂D_m are oriented in the counterclockwise direction. Since the jump of u on the unit cell along the x-direction is equal to 1, we get

$$\int_{\mathscr{D}_0} \frac{\partial u}{\partial x}\, dx\, dy = 1 - \sum_{m=1}^{N} \int_{D_m} \frac{\partial u}{\partial x}\, dx\, dy \tag{4.6}$$

Along similar lines we have

$$\int_{\mathscr{D}_0} \frac{\partial u}{\partial y}\, dx\, dy = -\sum_{m=1}^{N} \int_{D_m} \frac{\partial u}{\partial y}\, dx\, dy \tag{4.7}$$

Then (4.2)–(4.3) become

$$\lambda_e^x - i\lambda_e^{xy} = 1 + (\lambda_1 - 1) \sum_{m=1}^{N} \int_{D_m} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) dx\, dy$$

Using (2.7) and the identity[1]

$$\psi_m(z) = \varphi_m'(z) = \frac{\partial}{\partial x} [\Re \varphi_m(z)] - i \frac{\partial}{\partial y} [\Re \varphi_m(z)]$$

we obtain

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho \sum_{m=1}^{N} \int_{D_m} \psi_m(z)\, dx\, dy$$

Due to the mean value theorem for harmonic functions we have

$$\lambda_e^x - i\lambda_e^{xy} = 1 + 2\rho v \frac{1}{N} \sum_{m=1}^{N} \psi_m(a_m) \tag{4.8}$$

where $v = N\pi r^2$ is the concentration of the inclusions of conductivity λ_1. Since $\psi_m(z)$ has been calculated in Theorem 3.3, formula (4.8) provides an exact formula for $\lambda_e^x - i\lambda_e^{xy}$.

Remark 4.1. In Section 3 we put $\Phi(z) = c \equiv 0$. If we formally keep the constant c, then (4.8) for $\rho = 0$ ($\lambda_1 = 1$) becomes $\lambda_e^x - i\lambda_e^{xy} = 1 + c$. Thus c must be zero since for $\rho = 0$ we have the uniform material of the unit conductivity without the inclusions

Assume that our composite material is macroscopically isotropic. Then the effective tensor Λ_e is of the form $\Lambda_e = \lambda_e I$, where λ_e is the effective conductivity (scalar), I is the unit matrix. In this case (4.8) becomes

$$\lambda_e = 1 + 2\rho v \frac{1}{N} \sum_{m=1}^{N} \psi_m(a_m) \tag{4.9}$$

Using (3.10) we write λ_e in the form of the series in v:

$$\lambda_e = 1 + 2\rho v [A_0 + A_1 v + A_2 v^2 + \cdots] \tag{4.10}$$

where

$$A_p = \frac{1}{\pi^p N^{p+1}} \sum_{m=1}^{N} \psi_m^{(p)}(a_m), \qquad p = 0, 1, \ldots \tag{4.11}$$

We now use the recurrence relations from Theorem 3.4 to calculate the coefficients A_p in terms of the Eisenstein functions. For the sake of simplicity of the final formulas we apply the convention

$$E_p(z - a_k) := \begin{cases} E_p(z - a_k), & k \neq m \\ \sigma_p(z - a_m) & k = m \end{cases}$$

where the number m is the number of the corresponding function $\psi_m^{(p)}(z)$. In particular

$$E_p(a_k - a_m) := \begin{cases} E_p(a_k - a_m), & k \neq m \\ S_p & k = m \end{cases} \tag{4.12}$$

Here the functions E_p, σ_p and the constants S_p are derived in Appendix C. Moreover, we assume that all indexes $k, k_1, \ldots, k_s$ in the formulas below change from 1 to N. Then

$$\psi_m^{(0)}(z) = 1$$

$$\psi_m^{(1)}(z) = \rho \sum_k E_2(z - a_k)$$

$$\psi_m^{(2)}(z) = \rho^2 \sum_{k, k_1} \overline{E_2(a_k - a_{k_1})}\, E_2(z - a_k)$$

$$\begin{aligned} \psi_m^{(3)}(z) = & -2!\, \rho^2 \sum_{k, k_1} \overline{E_3(a_k - a_{k_1})}\, E_3(z - a_k) \\ & + \rho^3 \sum_{k, k_1, k_2} \overline{E_2(a_k - a_{k_1})}\, E_2(a_{k_1} - a_{k_2})\, E_2(z - a_k) \end{aligned} \tag{4.13}$$

$$\begin{aligned} \psi_m^{(4)}(z) = & \; 3!\, \rho^2 \sum_{k, k_1} \overline{E_4(a_k - a_{k_1})}\, E_4(z - a_k) \\ & - 2!\, \rho^3 \sum_{k, k_1, k_2} [\overline{E_3(a_k - a_{k_1})}\, E_2(a_{k_1} - a_{k_2})\, E_3(z - a_k) \\ & + \overline{E_3(a_k - a_{k_1})}\, E_3(a_{k_1} - a_{k_2})\, E_3(z - a_k)] \\ & + \rho^4 \sum_{k, k_1, k_2, k_3} \overline{E_2(a_k - a_{k_1})}\, E_2(a_{k_1} - a_{k_2})\, \overline{E_2(a_{k_2} - a_{k_3})}\, E_2(z - a_k) \end{aligned}$$

Substitute (4.13) in (4.11) and use the following notations

$$X_l := \sum_{k,\,k_1} E_l(a_k - a_{k_1}) = \sum_{k=1}^{N} \sum_{k_1 \neq k} E_l(a_k - a_{k_1}) + N\sigma_l(0) \tag{4.14}$$

$$X_{p_1 \cdots p_M} := \sum_{m,\,k_1,\ldots,\,k_M} E_{p_1}(a_m - a_{k_1})\, \overline{E_{p_2}(a_{k_1} - a_{k_2})} \cdots \mathbf{C}^{M-1} E_{p_M}(a_{k_{M-1}} - a_{k_M})$$

where $\mathbf{C}$ is the operator of complex conjugation. If M is odd, then $\mathbf{C}^{M-1}$ is the identity operator. If M is even, then $\mathbf{C}^{M-1}$ is the operator of complex conjugation. We also use the notation

$$\begin{aligned} Y_{p_1 \cdots p_M} := & \sum_{m=1}^{N} \sum_{k_1 \neq m}^{N} \cdots \sum_{k_M \neq k_{M-1}}^{N} E_{p_1}(a_m - a_{k_1}) \\ & \times \overline{E_{p_2}(a_{k_1} - a_{k_2})} \cdots \mathbf{C}^{M-1} E_{p_M}(a_{k_{M-1}} - a_{k_M}) \end{aligned} \tag{4.15}$$

to represent the convolutions $E_{p_1}(a_m - a_{k_1})\, \overline{E_{p_2}(a_{k_1} - a_{k_2})} \cdots \mathbf{C}^{M-1} E_{p_M}(a_{k_{M-1}} - a_{k_M})$ in an explicit form. We will need this notation when evaluating some integrals arising in the analysis of the "shaking" (random) configurations.

Then according to (4.12), (4.14) and (4.15)

$$\begin{aligned} X_{p_1 \cdots p_M} = & Y_{p_1 \cdots p_M} + S_{p_M} Y_{p_1 \cdots p_{M-1}} + S_{p_M} S_{p_{M-1}} Y_{p_1 \cdots p_{M-2}} \\ & + \cdots + S_{p_M} S_{p_{M-1}} \cdots S_{p_1} \end{aligned} \tag{4.16}$$

and we obtain

$$\begin{aligned} A_0 &= 1, \qquad A_1 = \frac{\rho}{\pi N^2} X_2, \qquad A_2 = \frac{\rho^2}{\pi^2 N^3} X_{22}, \\ A_3 &= \frac{1}{\pi^3 N^4} [\, -2\rho^2 X_{33} + \rho^3 X_{222}], \\ A_4 &= \frac{1}{\pi^4 N^5} [\, 3!\, \rho^2 X_{44} - 2!\, \rho^3 (X_{332} + X_{233}] + \rho^4 X_{2222}] \\ A_5 &= \frac{1}{\pi^5 N^6} [\, -4!\, \rho^2 X_{55} + 3!\, \rho^3 (X_{442} + X_{343} + X_{244}) \\ & \quad - 2!\, \rho^4 (X_{3322} + X_{2332} + X_{2233}) + \rho^5 X_{22222}] \\ A_6 &= \frac{1}{\pi^6 N^7} [\, 5!\, \rho^2 X_{66} - 4!\, \rho^3 (X_{255} + X_{354} + X_{453} + X_{552}) \\ & \quad + 3!\, \rho^4 (X_{2244} + X_{2343} + X_{3333} + X_{2442} + X_{3432} + X_{4422}) \\ & \quad - 4!\, \rho^5 (X_{22233} + X_{22332} + X_{23322} + X_{33222}) + \rho^6 X_{222222})] \end{aligned} \tag{4.17}$$

Here we have calculated $A_0, A_1, ..., A_6$. It is possible to proceed and to calculate the next terms $A_7, A_8, ...$. One can see that the coefficients A_p are determined via $X_{p_1 \cdots p_M}$, which are sums of the Eisenstein functions of the arguments of the type $a_k - a_m$ with some factors (e.g., $\rho/\pi N^2$). Hence, we have obtained the following result.

Theorem 4.2. Let the array of inclusions be such that the effective homogenized medium is isotropic. Then the effective conductivity has the form

$$\lambda_e = 1 + 2\rho v \sum_{p=0}^{6} A_p v^p + O(v^8) \tag{4.18}$$

Here A_p are calculated in (4.14) and (4.17), where E_p are the Eisenstein functions, N is the number of inclusions in the unit cell; ρ is the contrast parameter and the effective tensor Λ_e is defined in (4.1)–(4.3).

5. EFFECTIVE CONDUCTIVITY OF A RANDOM COMPOSITE

In this section we apply the formulas from Section 4 to evaluate the effective conductivity λ_e of an isotropic random composite material. For the sake of simplicity we choose $N = 4$ and consider the centers a_1, a_2, a_3, a_4 as i.i.d. random variables. Let

$$f_k(z) := \begin{cases} \dfrac{1}{\pi d^2}, & |z - b_k| < d \\ 0, & \text{otherwise} \end{cases}$$

be the probability density, which corresponds to the continuous uniform law, that is the point a_k is uniformly distributed inside the disk $|z - b_k| < d$. Here $b_1 = \frac{1}{4}(1+i)$, $b_2 = \frac{1}{4}(-1+i)$, $b_3 = \frac{1}{4}(-1-i)$, $b_4 = (1-i)$ are the sites of the regular square array. Thus the disks D_k are randomly distributed inside the unit periodicity cell (square) $Q_{(0,0)}$, so that we have the double periodic random structure. We assume that $d + r < \frac{1}{4}$ (see Fig. 1). This means that each disk lies in the prescribed quarter of the original square cell and does not cross or touch the coordinate axes. For the sake of simplicity we decompose each A_p into the sum $A_p = A_p^{(0)} + \tilde{A}_p$, where $A_p^{(0)}$ corresponds to the deterministic (periodic) structure and $\tilde{A}_p$ is the correction due to the randomness (the "fluctuation part"). Since (4.10) is linear in A_p we obtain the same decomposition for λ_e: $\lambda_e = \lambda_e^{(0)} + \tilde{\lambda}_e$. Calculating

the expected value from the latter relation we get $\langle \lambda_e \rangle = \lambda_e^{(0)} + \langle \tilde{\lambda}_e \rangle$. The deterministic part $\lambda_e^{(0)}$ corresponds to a single inclusion in a square periodicity cell and has been calculated by several authors (see refs. 17, 19, and 25). We now calculate the expectation of the fluctuation part

$$\langle \tilde{\lambda}_e \rangle = 2\rho v[\langle \tilde{A}_0 \rangle + \langle \tilde{A}_1 \rangle v + \langle \tilde{A}_2 \rangle v^2 + \cdots] \tag{5.1}$$

$$\langle \tilde{A}_p \rangle := \int_{[\Omega_{(0,0)}]^4} \tilde{A}_p(a_1, ..., a_4) \prod_{k=1}^{4} f_k(a_k)(d\sigma)^4, \qquad p = 0, 1, 2, ... \tag{5.2}$$

where $a_k = x + iy$ $(k = 1, 2, 3, 4)$ are the variables of integration, f_k is the probability density function, $d\sigma := dx\, dy$. Here we identify the random variable a_k with the parameter of integration. The coefficients $\tilde{A}_p(a_1, ..., a_4)$ are expressed through the special terms $J_{p_1 \cdots p_M}$ defined as follows:

$$J_{p_1 \cdots p_M} = \langle Z_{p_1 \cdots p_M} \rangle \tag{5.3}$$

where

$$Z_{p_1 \cdots p_M} = E_{p_1}(a_{k_0} - a_{k_1})\, \overline{E_{p_2}(a_{k_1} - a_{k_2})} \cdots \mathbf{C}^{M-1} E_{p_M}(a_{k_{M-1}} - a_{k_M})$$

First we calculate $J_{p_1 \cdots p_M}$, then $\langle X_{p_1 \cdots p_M} \rangle$ from (4.14)–(4.17). The details of these calculations are presented in Appendix D. In particular, we have proved in Appendix D that $J_{p_1 \cdots p_M}$ is an analytic function of d^2, and $\langle X_{p_1 \cdots p_M} \rangle$ can be computed as a sum of $J_{p_1 \cdots p_M}$. This implies that $\langle X_{p_1 \cdots p_M} \rangle$ is also analytic in d^2 and can be expanded into a convergent power series in d^2. Convergence in d^2 holds for admissible d^2, i.e., $d + r < \frac{1}{4}$, since the uniform convergence holds for all parameters in the method of successive approximations for the general functional equation (3.7). Hence we can decompose $\langle X_{p_1 \cdots p_M} \rangle$

$$\langle X_{p_1 \cdots p_M} \rangle = X^{(0)}_{p_1 \cdots p_M} + \langle \tilde{X}_{p_1 \cdots p_M} \rangle$$

Here the first term corresponds to the deterministic part $(d = 0)$ and the second term corresponds to the fluctuations due to the random "shaking." The solution $\psi_k(z)$ of (3.6) is analytic in a_k, since each term in the series (3.8) is analytic in a_k and the series uniformly converges in ρ (analytic in ρ). Therefore $\psi_k(a_k)$ is analytic in a_k. The latter implies the coefficients A_p are analytic in a_k.

To compute the dependence of the expected value $\langle A_p \rangle$ on the parameter d^2 we write $a_k = b_k + Re^{i\theta}$, $0 \leqslant R \leqslant d$ and integrate in R from 0

to d (for $d<\frac{1}{4}-r$). As a result of these calculations (see Appendix D) we get the following formulas for $\langle \tilde{A}_p \rangle$ $(p=0, 1, 2)$:

$$\begin{aligned} \langle \tilde{A}_0 \rangle &= \langle \tilde{A}_1 \rangle = 0 \\ \langle \tilde{A}_2 \rangle &= \frac{\rho^2}{\pi^2 4^3} \langle \tilde{X}_{22} \rangle = \frac{\rho^2}{16\pi^2} \sum_{l=1}^{\infty} h_{4l}^2 (4l-1)\, d^{8l-4} \end{aligned} \tag{5.4}$$

where h_{4l} is obtained by evaluating the Rayleigh sums (see Lemma 6.3). We can proceed to obtain analogous analytical formulas for $\langle \tilde{A}_p \rangle$ when $p>2$. One can see that each term $\langle \tilde{A}_p \rangle$ $(p=0, 1, 2)$ is non-negative. We next show that this is true for all p.

Theorem 5.1. The expected values of the fluctuation of the corrections defined in (5.2) are non-negative

$$\langle \tilde{A}_p \rangle \geqslant 0, \qquad p=0, 1, 2,... \tag{5.5}$$

Proof. The proof is based on the observation that the sign of $\langle Y_{p_1 \cdots p_M} \rangle$ (see (4.14)–(4.17)) is equal to $(-1)^{\chi(\mathbf{P})}$ with $\mathbf{P}=(p_1, p_2,..., p_M)$ ($\chi(\mathbf{P})$ is defined in (6.24)) and keeping track of the sign of each $\langle X_{p_1 \cdots p_M} \rangle$ and $\langle Y_{p_1 \cdots p_M} \rangle$ in A_p (see Lemma 6.5).

It follows from (3.11) and (4.11) that A_p can be constructed in the following way. First we determine

$$\begin{aligned} \frac{1}{N} \sum_{m=1}^{N} \psi_m^{(p)}(z) = \frac{1}{N} \sum_{m,k} [& \overline{\psi_{0k}^{(p-1)}} E_2(z-a_k) \\ & + \overline{\psi_{1k}^{(p-2)}} E_3(z-a_k) + \cdots + \overline{\psi_{p-2,k}^{(1)}} E_p(z-a_k)] \end{aligned}$$

Then we substitute $z=a_m$

$$\begin{aligned} A_p &= \frac{1}{N} \sum_{m=1}^{N} \psi_m^{(p)}(a_m) \\ &= \frac{1}{N} \sum_{m,k} [\overline{\psi_{0k}^{(p-1)}} E_2(a_m-a_k) \\ &\quad + \overline{\psi_{1k}^{(p-2)}} E_3(a_m-a_k) + \cdots + \overline{\psi_{p-2,k}^{(1)}} E_p(a_m-a_k)] \end{aligned} \tag{5.6}$$

We now apply the induction in p to check the following. If $\langle X_{p_1 \cdots p_M} \rangle > 0$ (<0) then it enters $\langle A_p \rangle$ with the plus (minus) sign and a positive factor respectively that is $\langle A_p \rangle$ computed by (5.6) is a linear combination of $|\langle X_{p_1 \cdots p_M} \rangle|$ with non-negative coefficients. Using (5.4) we see that this is

true for $\langle A_p \rangle$, $p = 0, 1, 2$. Now we assume that this is true for $s = 0, 1, 2, \ldots, p-1$ and consider $\langle A_p \rangle$ as given by (5.6). Take the term $\overline{\psi_{0k}^{(p-1)}}$ in (5.6). It contains the quantities $\langle X_{p_1 \cdots p_M} \rangle$. We multiply $\overline{\psi_{0k}^{(p-1)}} E_2(z - a_k)$ by $E_2(z - a_k)$, substitute $z = a_m$, sum up over all $m = 1, 2, \ldots, N$, divide by N and calculate the expectation $\langle\ \rangle$. Then each $\langle X_{p_1 \cdots p_M} \rangle$ becomes $\langle X_{p_1 \cdots p_M p_{M+1}} \rangle$, $p_{M+1} = 2$. Now take $\overline{\psi_{1k}^{(p-2)}}$ in (5.6). In the same way we see that $\langle X_{p_1 \cdots p_M} \rangle$ becomes $\langle X_{p_1 \cdots p_M p_M + 1 p_{M+1}} \rangle$, $p_{M+1} = 3$, moreover it is multiplied by (-1) in accordance with (6.24). On the other hand $\overline{\psi_{1k}^{(p-2)}}$ is also multiplied by (-1), since differentiation transforms the function $E_l(z)$ as follows $E_l'(z) = -lE_{l+1}(z)$. Then we again preserve the positive sign. Same argument applies to other terms in (5.6) and theorem is proved.

APPENDIX A

In the present section we prove the relation

$$[\overline{\varphi(t)}]' = -\left(\frac{r}{t - a_k}\right)^2 \overline{\varphi'(t)}, \qquad |t - a_k| = r \tag{6.1}$$

which is valid for each $\varphi(z)$ represented by the expansion

$$\varphi(z) = \sum_{l=0}^{\infty} \alpha_k (z - a_k)^l, \qquad |z - a_k| \leqslant r$$

We have $t = r^2/(\overline{t - a_k}) + a_k$ on $|t - a_k| = r$, hence the left-hand side of (6.1) can be written in the form

$$[\overline{\varphi(t)}]' = \sum_{l=1}^{\infty} \overline{\alpha_k} \frac{r^{2l}(-l)}{(t - a_k)^{l+1}} \tag{6.2}$$

The right-hand side of (6.1) can be written in the form

$$-\left(\frac{r}{t - a_k}\right)^2 \overline{\varphi'(t)} = -\left(\frac{r}{t - a_k}\right)^2 \sum_{l=1}^{\infty} l\,\overline{\alpha_k} \frac{r^{2(l-1)}}{(t - a_k)^{l-1}} \tag{6.3}$$

Comparison of (6.2) and (6.3) implies (6.1).

APPENDIX B

In ref. 26 the problem of (2.6) type is called the **R**-linear conjugation problem. If $\rho = 0$, $(\lambda_1 = 1)$, we get the **C**-linear problem, when φ and φ_k are

related to **C**-linear conjugation condition. If $\rho = 1$ ($\lambda_1 = \infty$) or $\rho = -1$ ($\lambda_1 = 0$), we arrive at the Hilbert–Riemann problem. In our case $|\rho| < 1$. Qualitative study of this case has been performed in ref. 22. We now show that the conjugation conditions (2.2) can be written in the form (2.6). The complex potentials have the following structure

$$\varphi(z) + z = u(z) + iv(z), \qquad z \in \mathscr{D}_0 \cup \partial\mathscr{D}_0$$

$$\varphi_k(z) = \frac{\lambda_1 + 1}{2}(u_k(z) + iv_k(z)), \qquad z \in D_k \cup T_k$$

where v and v_k are harmonic conjugate to u and u_k, respectively. Let $\partial/\partial s$ is the derivative in the natural parameter s. Using the Cauchy–Riemann equation[10] (Chapt. 5, p. 382) $\partial u/\partial n = -\partial v/\partial s$ and $\partial u_k/\partial n = -\partial v_k/\partial s$ we can write the second relation (2.2) in the form

$$\frac{\partial v}{\partial s} = \lambda_1 \frac{\partial v_k}{\partial s} \qquad \text{on } T_k \tag{6.4}$$

Integration (6.4) in s yields

$$v = \lambda_1 v_k + c_k \qquad \text{on } T_k$$

where c_k is an arbitrary real constant (see for example (2.3) and ref. 1). We put $c_k = 0$, since a complex potential is determined up to a purely imaginary additive constant. Hence (2.2) becomes

$$u = u_k, \quad v = \lambda_1 v_k \qquad \text{on } T_k \tag{6.5}$$

Add the first relation (6.5) and the second relation (6.5) multiplied by i. We have

$$u + iv = u_k + i\lambda_1 v_k \qquad \text{on } T_k \tag{6.6}$$

Substituting $u_k = [1/(\lambda_1 + 1)](\varphi_k + \overline{\varphi_k})$, $v_k = 1/(\lambda_1 + 1) \cdot (1/i)(\varphi_k - \overline{\varphi_k})$ and $u + iv = \varphi(t) + t$ in (6.6) we obtain (2.6).

In general case, a harmonic function in a multiply connected domain is represented as the real part of a multi-valued analytic functions,[10] Chapt. 5, p. 375). We explain now, why we take the single-valued function $\varphi(z)$ in $\mathscr{D}_0$. The function $\varphi(z)$ is single-valued in $\mathscr{D}_0$ if and only if the integral of $\partial v/\partial s$ along each circle T_k is zero

$$\int_{T_k} \frac{\partial v}{\partial s}\, ds = 0 \tag{6.7}$$

To verify (6.7) we apply the second relation (6.5) and the Cauchy–Riemann equation

$$\int_{T_k} \frac{\partial v}{\partial s}\, ds = \lambda_1 \int_{T_k} \frac{\partial v_k}{\partial s}\, ds = \lambda_1 \int_{T_k} \frac{\partial u_k}{\partial n}\, dS$$

The latter integral is equal to zero which follows from the harmonicity of u_k inside T_k and the Green's identity. Therefore, (6.7) holds and $\varphi(z)$ is single-valued in $\mathscr{D}_0$.

APPENDIX C. ELLIPTIC FUNCTIONS

This section contains the description of the basic elliptic functions, namely the Weierstrass' function $\mathscr{P}(z)$[13] and the Eisenstein's functions $E_m(z)$.[31]

First, we consider the lattice sums

$$S_{2n} := \sum_{m_1, m_2}{}' (m_1 + im_2)^{-2n} \tag{6.8}$$

introduced by Eisenstein[31] and applied to composite materials by Rayleigh.[17] In the sum $\sum'_{m_1, m_2}$ the integer numbers m_1, m_2 range from $-\infty$ to $+\infty$ except the case when $m_1^2 + m_2^2 = 0$ The theory of elliptic functions and ref. 13 provide the following formula

$$\mathscr{P}(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n-1)\, S_{2n} z^{2n-2} \tag{6.9}$$

The function $\mathscr{P}(z)$ is doubly periodic. The Eisenstein's functions[31] are introduced as follows

$$E_m(z) := \sum_{m_1, m_2} (z - m_1 - im_2)^{-m}$$

The Eisenstein E_2 and the Weierstrass function $\mathscr{P}$ are related by the identities

$$E_2(z) = \mathscr{P}(z) + S_2$$

Moreover, $E_l'(z) = -lE_{l-1}(z)$. Introduce the modified Eisenstein functions

$$\sigma_l(z) = E_l(z) - z^{-l}, \qquad l = 1, 2, \ldots$$

which are analytic in $Q_{(0,0)}$, and $\sigma_l(0) = S_l$, where $S_l = 0$ for odd l. For the square array $S_6 = S_{10} = S_{14} = \cdots = 0$.[17] The remaining sums are positive and can be easily calculated by recursive formulas of the elliptic function theory,[13] for instance, $S_2 = \pi$, $S_4 = 3.15121$, $S_8 = 4.25577$, $S_{12} = 3.93885$.

Recall that any meromorphic double periodic function (elliptic function) is a rational function of $\mathscr{P}(z)$ and $\mathscr{P}'(z)$.[13] Since our problem is double periodic it is convenient to represent it via the Eisenstein functions using the relation between E_l and $\mathscr{P}(z)$.

APPENDIX D. CALCULATION OF INTEGRALS AND SUMS

Lemma 6.1. Let $\mathscr{K}$ be the disk $\{w \in \mathbf{C} : |w - b_k| < d\}$ and

$$J^*_{pq} := \frac{1}{\pi d^2} \int_{\mathscr{K}} E_p(a_m - w)\, \overline{E_q(w - a_l)}\, d\sigma \tag{6.10}$$

then

$$J^*_{pq} = \sum_{s=0}^{\infty} (-1)^s\, C^{p-1}_{s+p-1} C^{q-1}_{s+q-1} E_{s+p}(a_m - b_k)\, \overline{E_{s+q}(b_k - a_l)}\, \frac{d^{2s}}{s+1} \tag{6.11}$$

Proof. We use the following formula[31]

$$E^{(s)}_p(z) := \frac{d^s}{dz^s} E_p(z) = (-1)^s\, p(p+1)\cdots(p+s-1)\, E_{p+s}(z) \tag{6.12}$$

Then using the Taylor expansion we obtain

$$\begin{aligned} E_p(a_m - w) &= (-1)^p \sum_{s=0}^{\infty} \frac{1}{s!} E^{(s)}_p(b_k - a_m)(w - b_k)^s \\ &= \sum_{s=0}^{\infty} C^{p-1}_{p+s+1} E_{p+s}(a_m - b_k)(w - b_k)^s \end{aligned} \tag{6.13}$$

Here we use the relation $E_p(z) = (-1)^p E_p(-z)$ from ref. 31 and (6.12). Analogously

$$\overline{E_q(w - a_l)} = \sum_{s=0}^{\infty} (-1)^s\, C^{q-1}_{q+s+1} \overline{E_{q+s}(b_k - a_l)} \tag{6.14}$$

Let s_1 and s_2 be integer numbers. Using the polar coordinates $w - b_k = Re^{i\theta}$ we calculate the integral

$$\frac{1}{\pi d^2} \int_{\mathscr{K}} (w-b_k)^{s_1} \overline{(w-b_k)}^{s_2} \, d\sigma = \begin{cases} 0, & \text{if} \quad s_1 \neq s_2 \\ \dfrac{d^{2s}}{s+1}, & \text{if} \quad s_1 = s_2 \end{cases} \tag{6.15}$$

Substituting (6.14) and (6.13) in the definition of J_{pq} (6.10) we obtain (6.11). The lemma is proved.

We now calculate the integral

$$J^*_{p_1 \cdots p_M} := \frac{1}{(\pi d^2)^{M-1}} \int_{\mathscr{K}_1} \cdots \int_{\mathscr{K}_{M-1}} E_{p_1}(a_{k_0} - a_{k_1}) \, \overline{E_{p_2}(a_{k_1} - a_{k_2})} \times \cdots \times \mathbf{C}^{M-1} E_{p_M}(a_{k_{M-1}} - a_{k_M}) (d\sigma)^{M-1}$$

which is a slight modification of $J_{p_1 \cdots p_M}$ defined in (5.3). Here $\mathbf{C}$ is the operator of complex conjugation; $a_{k_1}, a_{k_2}, \ldots, a_{k_{M-1}}$, are the variables of integration, a_{k_0} and a_{k_M} are parameters, $(d\sigma)^{M-1} := dx_1 \, dy_1 \cdots dx_{M-1} \, dy_{M-1}$, $a_{k_j} = x_j + iy_j$.

Lemma 6.2. We have

$$J^*_{p_1 \cdots p_M} = \sum_{s_1, \ldots, s_{M-1}} (-1)^{s_1 + \cdots + s_{M-1}} c[p, s] \, E_{p_1 + s_1}(a_{k_0} - b_{k_1}) \times \overline{E_{p_2 + s_1 + s_2}(b_{k_1} - b_{k_2})} \cdots \mathbf{C}^{M-1} E_{p_M + s_{M-1}}(b_{k_{M-1}} - a_{k_M}) \times \frac{d^{2(s_1 + \cdots + s_{M-1})}}{(s_1 + 1) \cdots (s_{M-1} + 1)} \tag{6.16}$$

where

$$c[p, s] := C^{p_1 - 1}_{p_1 + s_1 - 1} C^{p_2 - 1}_{p_2 + s_1 - 1} C^{p_2 + s_1 - 1}_{p_2 + s_1 + s_2 - 1} C^{p_3 - 1}_{p_3 + s_2 - 1} \cdots C^{p_M - 1}_{p_M + s_{M-1} - 1}$$

$$\sum_{s_1, \ldots, s_{M-1}} = \sum_{s_1 = 0}^{\infty} \sum_{s_2 = 0}^{\infty} \cdots \sum_{s_{M-1} = 0}^{\infty}$$

Proof. It is based on Lemma 6.1. For the sake of definiteness we choose M to be an even number. First we integrate in a_{k_1} in (6.16):

$$J^*_{p_1 \cdots p_M} = \frac{1}{(\pi d^2)^{M-2}} \int_{\mathscr{K}_2} \cdots \int_{\mathscr{K}_{M-1}} \sum_{s_1 = 0}^{\infty} C^{p_1 - 1}_{p_1 + s_1 - 1} C^{p_2 - 1}_{p_2 + s_1 - 1} \times E_{p_1 + s_1}(a_{k_0} - b_{k_1}) \, \overline{E_{p_2 + s_1}(b_{k_1} - a_{k_2})} \times E_{p_3}(a_{k_2} - a_{k_3}) \cdots \overline{E_{p_M}(a_{k_{M-1}} - a_{k_M})} (d\sigma)^{M-2}$$

where $a_{k_2},..., a_{k_{M-1}}$ are the variables of integration; a_{k_0}, a_{k_1}, a_{k_M} are parameters. Next we integrate in a_{k_2} and so on. After repeating this integration $M-2$ more times we obtain (6.16).

Lemma is proved.

We have integrated the convolution $E_{p_1}(a_{k_0}-a_{k_1})\,\overline{E_{p_2}(a_{k_1}-a_{k_2})}\cdots \mathbf{C}^{M-1}E_{p_M}(a_{k_{M-1}}-a_{k_M})$ in the variables $a_{k_1}, a_{k_2},..., a_{k_{M-1}}$ and have obtained $J^*_{p_1\cdots p_M}$. Next we need to calculate the integral in a_{k_0} and a_{k_M}. It is straightforward to check that the integration of $J^*_{p_1\cdots p_M}$ in a_{k_0} and a_{k_M} gives

$$\begin{aligned}
&\frac{1}{(\pi d^2)^{M+1}}\int_{\mathscr{K}_0}\cdots\int_{\mathscr{K}_M} E_{p_1}(a_{k_0}-a_{k_1})\,\overline{E_{p_2}(a_{k_1}-a_{k_2})}\\
&\quad\cdots \mathbf{C}^{M-1}E_{p_M}(a_{k_{M-1}}-a_{k_M})(d\sigma)^{M+1}\\
&=\frac{1}{(\pi d^2)^{M+1}}\int_{\mathscr{K}_1}\cdots\int_{\mathscr{K}_{M-1}} E_{p_1}(b_{k_0}-a_{k_1})\,\overline{E_{p_2}(a_{k_1}-a_{k_2})}\\
&\quad\cdots \mathbf{C}^{M-1}E_{p_M}(a_{k_{M-1}}-b_{k_M})(d\sigma)^{M-1} \qquad (6.17)
\end{aligned}$$

Hence, we have $J^*_{p_1\cdots p_M}=(1/\pi^2 d^4)\,J_{p_1\cdots p_M}$ when $a_{k_0}=b_{k_0}$ and $a_{k_M}=b_{k_M}$.

According to the theory of elliptic functions we introduce the parameters related to periods:

$$w_1=\frac{1}{2},\qquad w_2=-\frac{1+i}{2},\qquad w_3=\frac{i}{2} \qquad (6.18)$$

One can see that all arguments of $J_{p_1\cdots p_M}$ have the form b_p-b_q $(p\neq q)$, and therefore they can only take the values $\pm w_k$ $(k=1,2,3)$. It is known[19] that for the square array all values $E_p(w_k)$ are real, hence we can omit the complex conjugation in (6.16) for $a_{k_0}=b_{k_0}$, $a_{k_M}=b_{k_M}$.

Lemma 6.3. Let $h_p:=\sum_{k=1}^3 E_p(w_p)$, then $h_p=(2^p-1)\,S_p$, where S_p are the Rayleigh suns (see Appendix C).

Proof. It is based on the relation $\sum_{k=1}^3 E_p(w_p)+S_p=\hat{S}_p$, where $\hat{S}_p=\sum'_{m_1,m_2}(m_1/2+i(m_2/2))^{-p}=2^p\sum'_{m_1,m_2}(m_1+im_2)^{-p}=2^pS_p$ corresponds to a quarter of the original unit cell (see the definition (6.8) of the sum S_p).

This proves the lemma.

Using (4.16) we now are ready to calculate the following sums

$$\begin{aligned}
\langle X_{p_1\cdots p_M}\rangle &= \langle Y_{p_1\cdots p_M}\rangle + S_{p_M}\langle Y_{p_1\cdots p_{M-1}}\rangle + S_{p_M}S_{p_{M-1}}\langle Y_{p_1\cdots p_{M-2}}\rangle\\
&\quad + \cdots + S_{p_M}S_{p_{M-1}}\cdots S_{p_1}\\
\langle Y_{p_1\cdots p_M}\rangle &= \sum_{m=1}^{4}\sum_{k_1\neq M}\cdots\sum_{k_M\neq k_{m-1}} J_{p_1\cdots p_M}
\end{aligned}$$

where $\langle Y_{p_1 \cdots p_M} \rangle$ can also be written in the form

$$\langle Y_{p_1 \cdots p_M} \rangle = \sum_{s_1, \ldots, s_{M-1}} (-1)^{s_1 + \cdots + s_{M-1}} \times c[p, s]\, I_{p_1 + s_1,\, p_2 + s_1 + s_2 \cdots p_M + s_{M-1}} \frac{d^{2(s_1 + \cdots + s_{M-1})}}{(s_1 + 1) \cdots (s_{M-1} + 1)}$$

where

$$I_{p_1 \cdots p_M} = E_{p_1}(b_{k_0} - b_{k_1})\, \overline{E_{p_2}(b_{k_1} - b_{k_2})} \cdots \mathbf{C}^{M-1} E_{p_M}(b_{k_{M-1}} - b_{k_M})$$

Lemma 6.4. The following representation holds:

$$I_{p_1 \cdots p_M} = 4 h_{p_1} h_{p_2} \cdots h_{p_M} \tag{6.19}$$

Proof. It follows from the straightforward calculations. First we put $M = 1$, then

$$\eta_1 := E_p(b_1 - b_2) \sum_{l \neq 2} E_q(b_2 - b_l) + E_p(b_1 - b_3) \sum_{l \neq 3} E_q(b_3 - b_l) + E_p(b_1 - b_4) \sum_{l \neq 4} E_q(b_4 - b_l)$$

Furthermore η_1 contains the terms $E_p(w_{k_1})\, E_q(w_{k_2})$, which are equal to zero if p or q are odd. If p and q are even, then E_p and E_q are even functions and

$$\sum_{l \neq 2} E_q(b_2 - b_l) = E_q(b_2 - b_1) + E_q(b_2 - b_3) + E_q(b_2 - b_4) = \sum_{k=1}^{3} E_q(w_k) = h_q$$

Here we have used the relation $E_q(\bar{z}) = \overline{E_q(z)}$ and the condition that all $E_q(w_k)$ are real. The other sums in l give the same result. Then

$$\eta_1 = h_q(E_p(b_1 - b_2) + E_p(b_1 - b_3) + E_p(b_1 - b_4)) = h_p h_q$$

One can see that η_1 does not depend on m. Therefore, $I_{pq} = 4 h_p h_q$. Analogously we prove that

$$I_{p_1 \cdots p_M} = I_{p_1 \cdots p_{M-1}} h_{p_M}$$

By induction we conclude that (6.19) holds.

The lemma is proved.

Applying Lemma 6.4, we obtain the following explicit expression

$$\langle Y_{p_1 \cdots p_M} \rangle = \sum_{s_1, \ldots, s_{M-1}} (-1)^{s_1 + \cdots + s_{M-1}} c[p, s]\, h_{p_1 + s_1} h_{p_2 + s_1 + s_2} \cdots h_{p_M + s_{M-1}} \times \frac{d^{2(s_1 + \cdots + s_{M-1})}}{(s_1 + 1)(s_2 + 1) \cdots (s_{M-1} + 1)} \tag{6.20}$$

which is important in the investigation of the sign of each term

$$\langle \tilde{X}_{p_1 \cdots p_M} \rangle := \langle \tilde{Y}_{p_1 \cdots p_M} \rangle + S_{p_M} \langle \tilde{Y}_{p_1 \cdots p_{M-1}} \rangle + \cdots + S_{p_M} S_{p_{M-1}} \cdots S_{p_3} \langle \tilde{Y}_{p_1 p_2} \rangle \tag{6.21}$$

in $\tilde{A}_p$. The values $\langle \tilde{X}_{p_1 \cdots p_M} \rangle$, and hence $\langle \tilde{Y}_{p_1 \cdots p_M} \rangle$, correspond to the fluctuation part of the effective conductivity. The term $h_2 h_2 \cdots h_2$ enters only in the deterministic part (the coefficient in front of d^0, when $p_1 = \cdots = p_M = 2$ and $s_1 = \cdots = s_M = 0$ in (6.20)). Therefore, we do not care about the term $h_2 h_2 \cdots h_2$ and proceed now to investigate the remaining terms $h_{p_1 + s_1} h_{p_2 + s_1 + s_2} \cdots h_{p_M + s_{M-1}}$.

We observe that $\langle Y_{p_1 \cdots p_M} \rangle \neq 0$ iff

$$\begin{aligned} p_1 + s_1 &= 4l_1 \\ p_2 + s_1 + s_2 &= 4l_2 \\ &\vdots \\ p_{M-1} + s_{M-2} + s_{M-1} &= 4l_{M-1} \\ p_M + s_{M-1} &= 4l_M \end{aligned}$$

where $l_1, l_2, \ldots, l_M$ are integer numbers. Then

$$\begin{aligned} s_1 &= 4l_1 - p_1 \\ s_2 &= 4(l_2 - l_1) + p_1 - p_2 \\ &\vdots \\ s_{M-1} &= 4(l_{M-1} - l_{M-2} + \cdots) + P \\ s_{M-1} &= 4l_M - p_M \end{aligned} \tag{6.22}$$

where $P := (-1)^{M-1} (p_1 - p_2 + \cdots + (-1)^M p_{M-1})$. The first $M-1$ relations in (6.22) determine $l_1, l_2, \ldots, l_{M-1}$. The last relation in (6.22) can be

considered as the condition that $p_M + s_{M-1} = 4l_M$, $l_M \in \mathbf{Z}$. This condition implies that

$$\langle Y_{p_1 \cdots p_M} \rangle \neq 0 \qquad \text{iff} \quad P = 4l, \quad l \in \mathbf{Z} \tag{6.23}$$

Then

$$s_1 + s_2 + \cdots + s_{M-1} = \chi(\mathbf{P})$$

where $\mathbf{P} = (p_1, p_2, ..., p_M)$ and

$$\chi(\mathbf{P}) := \begin{cases} p_1 + p_3 + \cdots + p_{M-1} \pmod 2, & \text{if } M \text{ is even} \\ p_2 + p_4 + \cdots + p_{M-1} \pmod 2, & \text{if } M \text{ is odd} \end{cases}$$

Then $\langle Y_{p_1 \cdots p_M} \rangle$ has the sign $(-1)^{\chi(\mathbf{P})}$. Each non-zero term $\langle Y_{p_1 \cdots p_M} \rangle$ in $\langle A_p \rangle$ satisfies the condition that $p_1 + p_2 + \cdots + p_M$ is an even number. (6.23) implies that P is an even number too. Hence, for instance, for odd M we have

$$\begin{aligned} \chi(\mathbf{P}) &\equiv p_2 + p_4 + \cdots + p_{M-1} \pmod 2 \\ &\equiv (p_1 + p_2 + \cdots + p_M) - p_2 - p_4 - \cdots - p_{M-1} \pmod 2 \\ &\equiv p_1 + p_3 + \cdots + p_M \end{aligned}$$

Therefore, $\chi(\mathbf{P})$ can be defined as the sum of p_j with odd j:

$$\chi(\mathbf{P}) := \begin{cases} p_1 + p_3 + \cdots + p_{M-1} \pmod 2, & \text{if } M \text{ is even} \\ p_1 + p_3 + \cdots + p_M \pmod 2, & \text{if } M \text{ is odd} \end{cases} \tag{6.24}$$

We have demonstrated that the sign of $\langle \tilde{Y}_{p_1 \cdots p_M} \rangle$ is equal to $(-1)^{\chi(\mathbf{P})}$. Now we prove

Lemma 6.5. Let us fix an element $\mathbf{P} = (p_1, p_2, ..., p_M)$ and corresponding sets $(p_1, p_2, ..., p_v)$, $v = 2, 3, ..., M$, where the term $\langle \tilde{X}_{p_1 \cdots p_M} \rangle$ consists of the non-zero terms $\langle \tilde{Y}_{p_1 \cdots p_v} \rangle$ (see (6.21) and Appendix C). Then all terms $\langle \tilde{Y}_{p_1 \cdots p_v} \rangle$, $v = 2, 3, ..., M$, and hence $\langle \tilde{X}_{p_1 \cdots p_M} \rangle$ have the same sign $(-1)^{\chi(\mathbf{P})}$.

Proof. We have that $S_l \neq 0$ for the square array only if $l = 2, 4, 8, 12, ...$ (see Appendix C). Then (6.21) implies that each of the terms $\langle \tilde{X}_{p_1 \cdots p_M} \rangle$ consists of the terms $\langle \tilde{Y}_{p_1 \cdots p_v} \rangle$ with positive multipliers $S_{p_M} \cdots S_{p_{v+1}}$ only if $p_{v+1}, ..., p_M$ take the even values $2, 4, 8, 12, ...$. Then all non-zero terms in (6.21) have the same sign $(-1)^{\chi(\mathbf{P})}$.

The lemma is proved.

CONCLUSION

We first compute higher order terms in the Clausius–Mossotti formula, which is the expansion of the effective conductivity in the concentration of the fibers. This is done for an arbitrary number of unidirectional fibers, which are arbitrary located in the unit periodicity cell. Using the complex analysis techniques, we obtain analytical formulas for the coefficients in this expansion.

Next we use these results to study random arrays ("shaking" geometries) when the fibers are allowed to move randomly inside the periodicity cell according certain uniform distribution. We have computed analytical dependence of the coefficients in the Clausius–Mossotti formula on the distribution parameter d. Analysis of this dependence (Theorem 5.1) shows that the periodic array of the fibers has lower effective conductivity than any array obtained by the random shaking of the fibers. The latter is rigorously proved in the case of 4 cylinders in the unit cell and generalization for an arbitrary number of fibers is straightforward (but cumbersome).

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